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SYMETRIZATION OF VLASOV-POISSON EQUATIONS*

BRUNO DESPRÉS†

Abstract. We detail the spectrum of the linearized Vlasov-Poisson equation, and construct an original integro-differential operator which is related to the eigenstructure. It gives a new representation formula for the electric field, and yields new estimates for the linear Landau damping. Then we apply the technique to a problem with a dependence to the Debye length, and show weaker damping for small Debye length. A non linear variant of the main quadratic framework is finally discussed.

Key words. Vlasov-Poisson equation, Landau damping, integro-differential operator, Debye length.

1. Introduction. In this work we study various symetrized formulations of the Vlasov-Poisson equation and related equations. It will be performed mostly for the linearized versions of the non linear equations, even if we will consider the non linear problem at the end of this work.

The first model problem that we consider is a Vlasov equation for electrons in dimension one with periodic boundary conditions, that is $x \in I = [0, 2\pi]_{\text{per}}$. The ions are fixed, have total mass equal to $\sqrt{2\pi}$, and have vanishing current. With these hypothesis a first model problem writes

$$(1.1) \quad \begin{cases} \partial_t f + v \partial_x f - E \partial_v f = 0, & t > 0, \quad (x, v) \in I \times \mathbb{R}, \\ \partial_x E = \sqrt{2\pi} - \int_{\mathbb{R}} f dv, & t > 0, \quad x \in I. \end{cases}$$

The equation on E is the Gauss equation. It can be checked that it is compatible with the Ampère equation $\partial_t E = \int v f dv$ if the total momentum vanishes $\int_I \int_{\mathbb{R}} f v dv = 0$, which means that the total electric current vanishes: this is a reasonable assumption with respect to the physics. We deduce an alternative Vlasov-Ampère formulation

$$(1.2) \quad \begin{cases} \partial_t f + v \partial_x f - E \partial_v f = 0, & t > 0, \quad (x, v) \in I \times \mathbb{R}, \\ \partial_t E = \int_{\mathbb{R}} v f dv, & t > 0, \quad x \in I, \end{cases}$$

which the one used in this work. The equivalence between (1.1) and (1.2) is true if the two conditions are fulfilled at initial time

$$(1.3) \quad \begin{cases} \partial_x E = \sqrt{2\pi} - \int f dv & \text{at } t = 0 \text{ and } \forall x \in I, \\ \int_I \int_{\mathbb{R}} f v dv = 0, & \text{at } t = 0, \end{cases}$$

which is considered as granted in this work. The solutions of the Vlasov-Poisson equation satisfy the conservation of the physical energy

$$(1.4) \quad \frac{d}{dt} \left(\int_I \int_{\mathbb{R}} f(t, x, v) v^2 dv dx + \int_I E(t, x, v)^2 dx \right) = 0$$

and the boundedness of the density

$$(1.5) \quad 0 \leq f(t, x, v) \leq \|f_0\|_{L^\infty(I \times \mathbb{R})}, \quad f(0, x, v) = f_0(x, v).$$

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†LJLL, UPMC, 75252 Paris Cedex

Let us consider the normalized Maxwellian profile $G(v) = \exp(-\frac{v^2}{2})$ so that $\int G(v)dv = \sqrt{2\pi}$. The linearization of (1.3) for f close to a Maxwellian, namely $f = G + g$, yields

$$(1.6) \quad \begin{cases} \partial_t g + v\partial_x g + EvG = 0, & t > 0, \quad (x, v) \in I \times \mathbb{R}, \\ \partial_t E = \int_{\mathbb{R}} v g dv, & t > 0, \quad x \in I. \end{cases}$$

The Gauss relation reads now

$$(1.7) \quad \partial_x E = - \int_{\mathbb{R}} g dv.$$

The starting point of the analysis is the fact that (1.6) is endowed with a L^2 conservation property which writes

$$(1.8) \quad \frac{d}{dt} \left(\int_I \int_{\mathbb{R}} \frac{g^2}{G} dv dx + \int_I E^2 dx \right) = 0.$$

This conservation law is an indication that symetrization is possible, and that the solutions can be expressed as the action of a linear semi-group. The rest of this work is devoted to the consequences of such a structure, that will also be extended to non Maxwellian equilibrium function, provided they have one bump, that is they possess one and only one maximum point. The linear semi-group will be calculated in full details, and so is an extension of the method used in [17, 4, 5]. It yields a complementary approach to the seminal method of Landau [11] for which the modern treatment is to be found in [13]. We also apply the tools developed in this work to the non linear Vlasov-Poisson equation [6, 15] for ions

$$(1.9) \quad \begin{cases} \partial_t f + v\partial_x f + E\partial_v f = 0, & t > 0, \quad (x, v) \in I \times \mathbb{R}, \\ \lambda_D^2 \partial_x E = \int_{\mathbb{R}} f dv - n_e, & t > 0, \quad x \in I. \end{cases}$$

Here n_e is the density of electrons $E = -\partial_x \log n_e$, and $\lambda_D > 0$ is the Debye length. The interesting physical regime is when λ_D is small. The formal limit $\lambda_D \rightarrow 0^+$ equation has been called the Vlasov-Dirac equation [2]

$$\partial_t f + v\partial_x f - \partial_x \log \left(\int_{\mathbb{R}} f dv \log \right) \partial_v f = 0, \quad t > 0, \quad (x, v) \in I \times \mathbb{R}$$

known to be singular or close to be singular [8].

The most original object that is constructed in this work is a new integro-differential operator, denoted as L in the following, which has the ability to transform any solution of (1.6) into a solution of the transport equation in free space, that is without electric field. The foundations of the method come from a detailed calculation of the generalized eigenvectors of the linear semi-group attached to the energy identity (1.8).

This integro-differential operator allows an explicit calculation of the decay in time of the electric field in function of the regularity of the initial data. We obtain for example that the solution of (1.6) satisfies for $t \rightarrow \infty$

$$(1.10) \quad \|E(t)\|_{L^\infty(I)} \leq \frac{C_n}{t^n}, \quad \text{provided } \frac{g_0}{G^{\frac{1}{2}}} \in L^2(I : H^n(\mathbb{R})), \quad n \geq 1,$$

where the constant C_n is independent of the initial data. The extension of this estimate to the linearized version of (1.9) is

$$\|E(t)\|_{L^\infty(I)} \leq \frac{D_n}{t^n}, \quad \text{provided } \frac{g_0}{G^{\frac{1}{2}}} \in L^2(I : H^n(\mathbb{R})), \quad n \geq 2,$$

where the constant D_n is independent of the Debye length λ_D if $n \geq 2$ which is more restrictive than in (1.10). We therefore observe that a small Debye length results in a slightly weaker damping of the electric field for low regularity initial data.

This work is organized as follows. Our first section is devoted to the construction of the eigenvectors of the linearized Vlasov-Ampère equation around Maxwellians. The next section uses this construction to design a certain operator for more general equilibrium functions. It will be applied to obtain new formulas for the damping rate of the Landau damping in the linear regime. Next we extend some of the results to the non linear Vlasov-Poisson equation (1.9) and detail the sensibility of the asymptotic damping with respect to the Debye length. Finally we study the quadratic stability of the non linear Vlasov-Ampère equation which gives a more general setting in which the tools developed in this work can be employed.

Properties of the Hilbert transform, used everywhere throughout this work, can be found in [16].

2. Symetrization of the linearized system (1.6). In this section we continue for convenience to work with Maxwellian profiles. This hypothesis will be relaxed in the next section.

The square root of the Maxwellian is itself a Maxwellian

$$M(v) = \sqrt{G(v)} = \exp\left(-\frac{v}{4}\right).$$

Let us define the function $u = \frac{g}{M}$ which is solution of

$$(2.1) \quad \begin{cases} \partial_t u + v \partial_x u &= -v M E, & t > 0, & (x, v) \in I \times \mathbb{R}, \\ \partial_t E &= \int_{\mathbb{R}} u v M dv, & t > 0, & x \in I, \end{cases}$$

with the energy identity $\frac{d}{dt} \left(\int_I \int_{\mathbb{R}} u^2 dv dx + \int_I E^2 dx \right) = 0$.

2.1. Notations. To continue the analysis, we introduce the Hermite polynomials $H_n(v)$ which are orthonormal with respect to the Maxwellian weight $G(v)$, see [1].

- The Hermite polynomials admit the representation

$$H_n(v) = (-1)^n G(v)^{-1} \frac{d^n}{dv^n} G(v).$$

- The degree of H_n is n . The parity of H_n is the parity of n .
- Hermite polynomials are orthogonal with respect to the Maxwellian weight

$$\int H_n(v) H_m(v) G(v) dv = (2\pi)^{\frac{1}{2}} n! \delta_{nm}, \quad n, m \in \mathbb{N}.$$

- One has the recursion formula

$$H_{n+1}(v) = v H_n(v) - H_{n-1}(v).$$

- The family of Hermite polynomials is a Hilbert basis of the space of functions such that $\int_{\mathbb{R}} f^2(v) G(v) dv < \infty$.

Let us define for convenience $I_n(v) = (2\pi)^{-\frac{1}{4}} n!^{-\frac{1}{2}} H_n(v)$ and the Hermite functions which constitute a Hilbert basis of $L^2(\mathbb{R})$

$$(2.2) \quad \psi_n(v) = I_n(v) M(v).$$

The family $(\psi_n)_{n \in \mathbb{N}}$ is by construction orthonormal: $\int_{\mathbb{R}} \psi_p(v) \psi_q(v) dv = \delta_{pq}$. The first terms of the series are

$$\psi_0(v) = (2\pi)^{-\frac{1}{4}} M(v), \quad \psi_1(v) = (2\pi)^{-\frac{1}{4}} v M(v), \quad \psi_2(v) = (8\pi)^{-\frac{1}{4}} (v^2 - 1) M(v).$$

The recursion formula becomes after rescaling

$$(2.3) \quad \sqrt{n+1} \psi_{n+1}(v) = v \psi_n(v) - \sqrt{n} \psi_{n-1}(v), \text{ for all } n.$$

The system (2.1) is therefore rewritten as

$$(2.4) \quad \begin{cases} \partial_t u + v \partial_x u &= -\alpha \psi_1 E, & t > 0, & (x, v) \in I \times \mathbb{R}, \\ \partial_t E &= \alpha \int_{\mathbb{R}} u \psi_1 dv, & t > 0, & x \in I, \end{cases}$$

where $\alpha = (2\pi)^{\frac{1}{4}} > 0$ is a given real number which comes from our normalization of the mass of ions.

Assuming in view of the energy identity that $u(t) \in L^2(I \times \mathbb{R})$, we define the moments $\alpha_n(t) \in L^2(I)$ by

$$u(t, x, v) = \sum_n \alpha_n(t, x) \psi_n(v), \quad \alpha_n = \int_{\mathbb{R}} u \psi_n dv.$$

By construction $\|u\|_{L^2(I \times \mathbb{R})}^2 = \sum_{n \in \mathbb{N}} \|\alpha_n\|_{L^2(I)}^2$. We now construct the vector $U(t) = (E(t), \alpha_0(t), \alpha_1(t), \alpha_2(t), \dots)^t \in (L^2(I))^{\mathbb{N}}$. A natural space is $\ell^2 = \{w = (w_n), \sum_{\mathbb{N}} |w_n|^2 < \infty\}$ equipped with the norm

$$\|w\|_{\ell^2} = \sqrt{\sum_{\mathbb{N}} |w_n|^2}.$$

The Fourier transform in space will be denoted as $\widehat{w}_k = \widehat{w}(k) = \int_I e^{-ikx} w(x) dx$ for $k \in \mathbb{Z}$. The inverse formula is

$$(2.5) \quad w(x) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{ikx} \widehat{w}(k).$$

Other natural norms used throughout this work are for example

$$\|u\|_{L^2(I \times \mathbb{R})}^2 = \int_I \int_{\mathbb{R}} |u(x, v)|^2 dx dv = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{N}} |\widehat{u}_n(k)|^2$$

and

$$\|u\|_{L^2(I; H^n(\mathbb{R}))}^2 = \sum_{p=0}^n \int_I \int_{\mathbb{R}} \left| \frac{\partial^p}{\partial v^p} u(x, v) \right|^2 dx dv = \frac{1}{2\pi} \sum_{p=0}^n \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \left| \frac{\partial^p}{\partial v^p} \widehat{u}_k(v) \right|^2 dv.$$

2.2. Matrix formulation. The system (2.4) can be rewritten as

$$\partial_t U + A \partial_x U = -i B U$$

where $A \in \mathbb{R}^{\mathbb{N} \times \mathbb{N}}$ and $B \in \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ are sparse hermitian matrices

$$(2.6) \quad A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 2^{\frac{1}{2}} & 0 & \dots \\ 0 & 0 & 2^{\frac{1}{2}} & 0 & 3^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 3^{\frac{1}{2}} & 0 & 4^{\frac{1}{2}} \\ \dots & \dots & \dots & \dots & 4^{\frac{1}{2}} & \dots \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 & -i\alpha & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ i\alpha & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

This is evident from (2.3-2.4). After Fourier reduction it becomes

$$\partial_t \widehat{U}(k) = -iA_k \widehat{U}(k),$$

where the matrix of the system is $A_k = kA + B$

$$A_k = \begin{pmatrix} 0 & 0 & -i\alpha & 0 & 0 & \dots \\ 0 & 0 & k & 0 & 0 & \dots \\ i\alpha & k & 0 & k2^{\frac{1}{2}} & 0 & \dots \\ 0 & 0 & k2^{\frac{1}{2}} & 0 & k3^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & k3^{\frac{1}{2}} & 0 & k4^{\frac{1}{2}} \\ \dots & \dots & \dots & \dots & k4^{\frac{1}{2}} & \dots \end{pmatrix}$$

We refer to [9] for a similar formulation with moments.

REMARK 1. *The matrix A_k is sparse, hermitian for $k \in \mathbb{Z}$. Since A is an unbounded matrix, A_k defines an unbounded operator for $k \neq 0$. The matrix B is a finite rank perturbation. This fact will be of great help for the determination of the spectrum of A_k by means of a perturbation method.*

The solution of the equation $\partial_t \widehat{U}(k) = -iA_k \widehat{U}(k)$ is given by representation formula

$$(2.7) \quad \widehat{U}_k(t) = e^{-iA_k t} \widehat{U}_k(0).$$

Nevertheless $A_k \notin \mathcal{L}(l^2)$, so special care has to be devoted to the definition of the exponential of the matrix

$$(2.8) \quad e^{-iA_k t} = \sum_{n \in \mathbb{N}} \frac{(-it)^n}{n!} A_k^n.$$

We will rely on two different methods. The first one is based on a direct elementary proof of the convergence of the series (2.8). The second one is related to the computation of the spectrum of A_k .

2.3. Convergence of the series. The subspace of $X_p \subset l^2$ of series with compact support less than $p+1$ is

$$X_p = \{v = (v_n) \in l^2; v_n = 0 \text{ for } n \geq p+1\}, \quad p \in \mathbb{N}.$$

One has $A_k X_p \subset X_{p+1}$ which shows that

$$A_k \in \mathcal{L}(X_p, X_{p+1}) \quad p \in \mathbb{N}.$$

That is A_k can be seen as a bounded operator in $\mathcal{L}(X_p, X_{p+1})$ still equipped with the quadratic norm.

PROPOSITION 2. *One has*

$$(2.9) \quad \|A\|_{\mathcal{L}(X_p, X_{p+1})} \leq 2\sqrt{p}, \quad \forall p.$$

Proof. Let us denote by $\|\cdot\|_1$ the associated l^1 norm and by $\|\cdot\|_\infty$ the associated l^∞ norm. We proceed by interpolation between the l^1 , l^2 and l^∞ norms using

$$\|A\|_{\mathcal{L}(X_p, X_{p+1})} \leq \sqrt{\|A\|_1 \|A\|_\infty}.$$

Classical formula shows that the norm of an hermitian matrix $M = M^*$ of size N is

$$\|M\|_1 = \|M\|_\infty = \sum_{1 \leq i \leq N} \sum_{1 \leq j \leq N} |m_{ij}|.$$

The first case for $p = 0$ is trivial $\|A\|_1 = \|A\|_\infty = 0$. The general case is $p > 0$: one gets

$$\|A\|_1 = \|A\|_\infty = \sqrt{p-1} + \sqrt{p} \leq 2\sqrt{p}.$$

It yields the result. \square

PROPOSITION 3. *There exists a constant $C(k) \in \mathbb{N}$ such that*

$$(2.10) \quad \|A_k^n\|_{\mathcal{L}(X_p, V)} \leq \left(\frac{4|k|}{\alpha}\right)^n \sqrt{n^p n!}, \quad \forall p \geq C(k).$$

Proof. A consequence of the previous proposition is that $\|A_k\|_{\mathcal{L}(X_p, X_{p+1})} \leq \alpha + 2kp^{\frac{1}{2}}$. By iteration

$$\|A_k^n\|_{\mathcal{L}(X_p, l^2)} \leq \Pi_{q=0}^{n-1} (\alpha + 2|k|\sqrt{p+q}), \quad n \geq 1,$$

that is

$$\|A_k^n\|_{\mathcal{L}(X_p, l^2)} \leq \alpha^n \Pi_{q=0}^{n-1} (1 + \beta\sqrt{p+q}), \quad \beta = \frac{2|k|}{\alpha}.$$

Let $C(k)$ be the maximal entire such that $\beta\sqrt{C(k)} \leq 1$. Assume $C(k) < p$. Then $1 < \beta\sqrt{p} \leq \beta\sqrt{p+q}$ for non negative q . So

$$\Pi_{q=0}^{n-1} (1 + \beta\sqrt{p+q}) \leq 2^n \beta^n \Pi_{q=0}^{n-1} \sqrt{p+q} = 2^n \beta^n \sqrt{\frac{(n+p-1)!}{(p-1)!}}.$$

A crude bound yields the result and ends the proof. \square

The estimate (2.10) implies the normal convergence of the series (2.8) in the space $\mathcal{L}(X_p, X)$. That is $e^{-iA_k t} v_p$ is well defined in l^2 for all $v_p \in X_p$, and for all p .

PROPOSITION 4. *The exponential of the matrix (2.8) is well defined in $\mathcal{L}(X)$.*

Proof. This is standard, we refer to [3] for complementary details.

Since $-iA_k$ is an anti-hermitian matrix, one has that the usual property $\|e^{-iA_k t} v_p\| = \|v_p\|$ for all $v_p \in X_p$. Any function $v \in X$ can be approximated in X as close as needed by a sequence of function $v_p \in X_p$, that is $v_p \rightarrow v$ in X with $v_p \in X_p$. It defines $e^{-iA_k t} v$ as the limit in l^2 of the Cauchy series

$$e^{-iA_k t} v = \lim_{p \rightarrow \infty} e^{-iA_k t} v_p.$$

\square

2.4. The spectrum of A_k . The matrix A_k is a non bounded operator so may have continuous spectrum as well [10]. Any non zero vector $U \in \mathbb{C}^{\mathbb{N}}$ such that

$$A_k U = \lambda U, \quad \lambda \in \mathbb{C},$$

is called an eigenvector. Eigenvectors in the discrete spectrum belong to l^2 : they are the classical eigenvectors. Eigenvectors in the continuous spectrum are not in l^2 : they may be called generalized eigenvectors.

A key observation is that A_k is designed by adding a finite rank perturbation to kA . That is why we will first compute the eigenvectors of A , and after that use a perturbation method to compute the eigenvectors of A_k . This is a standard approach in scattering theory. We will use the notation that

$$e_p = (0, \dots, 0, 1, 0, \dots)^t$$

where the coefficient 1 is in the p th position.

PROPOSITION 5 (First step). *The spectrum of A is $S(A) = S_d(A) \cup S_c(A)$ where $S_d(A) = \{0\}$: there is one eigenvector $u_0 = e_1$ associated to the eigenvalue 0, $S_c(A) = \mathbb{R}$: the generalized eigenvector $U_\lambda = (0, \psi_0(\lambda), \psi_1(\lambda), \psi_2(\lambda), \dots)^t$ is in the continuous spectrum for all $\lambda \in \mathbb{R}$.*

These eigenvectors are complete: for all $U \in l^2$, one has the representation formula

$$U = (U, u_0)u_0 + \int_{\mathbb{R}} (U, U_\lambda) U_\lambda d\lambda$$

with

$$(2.11) \quad \|U\|^2 = |(U, u_0)|^2 + \int_{\mathbb{R}} |(U, U_\lambda)|^2 d\lambda.$$

Proof. First $u_0 = e_1$ is of course an eigenvector. Second the recursion formula (2.3) shows that U_λ is indeed an eigenvector for the eigenvalue λ . The two last formulas come from the fact that the family of Hermite functions $(\psi_n(\lambda))_{n \in \mathbb{N}}$ is an Hilbert basis of $L^2(\mathbb{R})$. \square

REMARK 6 (Spectrum of A_k for $k = 0$). *For $k = 0$, the spectrum of $A_0 = B$ is fully discrete. It is made of $\{\pm\alpha\}$ which are the eigenvalues of B and $\{0\}$ associated to an eigenspace of codimension 2.*

We now turn to the determination of the spectrum of A_k for $k \neq 0$. By comparison with the spectrum of A , it is reasonable to expect one eigenvalue in the discrete spectrum and a continuous spectrum. The eigenvalue in the discrete spectrum is easy to identify. Indeed

$$(2.12) \quad v_0^k = (k, -i\alpha, 0, 0, \dots)^t \in l^2$$

is associated the eigenvalue 0 which is therefore in the discrete spectrum. The continuous spectrum is more involved. A natural idea is to "modify", in a sense to be explained, the eigenvectors of kA to obtain those of A_k . We develop a priori such eigenvectors using the basis of eigenvectors of A , that is

$$(2.13) \quad V_\lambda^k = \mathbf{a}_\lambda^k u_0 + \int_{\mathbb{R}} \mathbf{f}_\lambda^k(\mu) U_\mu d\mu$$

where $\alpha_\lambda^k \in \mathbb{C}$ a complex number and \mathbf{f}_λ^k a function to be determined. Let us now discuss the type of "function" \mathbf{f}_λ^k that we need to consider. Since we develop a perturbation method, we desire to have the possibility to recover the continuous spectrum of A in case B is set to zero. In other words we desire that $V_\lambda^k = U_\lambda$ can be obtain in (2.13) with a certain "function": it is therefore necessary to assume that \mathbf{f}_λ^k could be a Dirac mass δ_λ . To treat the general case, we add principal values which can be considered as some continuous approximations of Dirac masses. So we will consider representations like

$$\mathbf{f}_\lambda^k(\mu) = \mathbf{b}_\lambda^k \delta_\lambda + \frac{\mathbf{c}_\lambda^k(\mu)}{\mu - \lambda}, \quad \text{where } \beta_\lambda^k \in \mathbb{C} \text{ and } \gamma_\lambda^k \text{ a function.}$$

Such a representation is very common for special solutions of transport equation. Actually this representation is the same as the one used in the normal mode method of Van Kampen [17, 4].

In summary we look for two complex numbers $\mathbf{a}_k^\lambda, \mathbf{b}_k^\lambda \in \mathbb{C}$ and for a smooth function \mathbf{c}_k^λ such that

$$V_\lambda^k = \mathbf{a}_\lambda^k e_1 + \mathbf{b}_k^\lambda U_\lambda + P.V. \int_{\mathbb{R}} \frac{\mathbf{c}_\lambda^k(\mu)}{\mu - \lambda} U_\mu d\mu$$

is an eigenvector of A_k associated to the eigenvalue $k\lambda$. The eigenvalue equation $A_k V_\lambda^k = \lambda k V_\lambda^k \iff k A V_\lambda^k + B V_\lambda^k = \lambda k V_\lambda^k$ writes

$$\begin{aligned} & \lambda k \mathbf{b}_k^\lambda U_\lambda + P.V. \int_{\mathbb{R}} k \mu \frac{\mathbf{c}_\lambda^k(\mu)}{\mu - \lambda} U_\mu d\mu + B \left(\mathbf{a}_\lambda^k e_1 + \mathbf{b}_k^\lambda U_\lambda + P.V. \int_{\mathbb{R}} \frac{\mathbf{c}_\lambda^k(\mu)}{\mu - \lambda} U_\mu d\mu \right) \\ (2.14) \quad & = \lambda k \mathbf{a}_\lambda^k e_1 + \lambda k \mathbf{b}_k^\lambda U_\lambda + P.V. \int_{\mathbb{R}} \lambda k \frac{\mathbf{c}_\lambda^k(\mu)}{\mu - \lambda} U_\mu d\mu. \end{aligned}$$

It simplifies into

$$(2.15) \quad k \int_{\mathbb{R}} \mathbf{c}_\lambda^k(\mu) U_\mu d\mu + B \mathbf{a}_\lambda^k e_1 + B \left(\mathbf{b}_k^\lambda U_\lambda + P.V. \int_{\mathbb{R}} \frac{\mathbf{c}_\lambda^k(\mu)}{\mu - \lambda} U_\mu d\mu \right) = \lambda k \mathbf{a}_\lambda^k e_1.$$

Equation (2.15) can be decomposed in two parts.

The first part is all equations in this infinite vectorial system, except the first one. We remark that $B U_\mu$ is colinear to $e_1 = u_0$ for all μ , and that $B e_1$ also belongs to e_1^\perp . One obtains a reduced infinite set of equations in the orthogonal of e_1 (e_1^\perp) of codimension one

$$k \int_{\mathbb{R}} \mathbf{c}_\lambda^k(\mu) U_\mu d\mu + B \mathbf{a}_\lambda^k e_1 = 0.$$

The definition of ψ_1 and of the U_μ implies that

$$B e_1 = i \alpha e_3 = i \alpha \int_{\mathbb{R}} \psi_1(\mu) U_\mu d\mu.$$

One obtains

$$(2.16) \quad k \mathbf{c}_\lambda^k(\mu) + i \alpha \psi_1(\mu) \mathbf{a}_\lambda^k = 0.$$

We now must add one scalar equation. It could be the first equation of the infinite system (2.15), which means taking the hermitian product of (2.15) against e_1 . However we prefer to use the orthogonality relation $(V_\lambda^k, v_0^k) = 0$ since the calculations are simpler (it can be checked to it is equivalent to the first equation of the infinite system). One obtains

$$\mathbf{a}_\lambda^k(e_1, v_0^k) + \mathbf{b}_k^\lambda(U_\lambda, v_0^k) + P.V. \int_{\mathbb{R}} \frac{\mathbf{c}_\lambda^k(\mu)}{\mu - \lambda} (U_\mu, v_0^k) d\mu = 0,$$

that is

$$0 = k\alpha_k^\lambda + \mathbf{b}_k^\lambda i\alpha\psi_0(\lambda) + i\alpha P.V. \int_{\mathbb{R}} \frac{\mathbf{c}_\lambda^k(\mu)}{\mu - \lambda} \psi_0(\mu) d\mu.$$

Elimination of the unknown function \mathbf{c}_λ^k using (2.16) yields

$$0 = \left(k^2 + \alpha^2 P.V. \int_{\mathbb{R}} \frac{\psi_1(\mu)\psi_0(\mu)}{\mu - \lambda} d\mu \right) \mathbf{a}_\lambda^k + i\alpha\psi_0(\lambda)k\mathbf{b}_k^\lambda$$

or after simplifications by α

$$(2.17) \quad 0 = \left(k^2 + P.V. \int_{\mathbb{R}} \frac{\mu}{\mu - \lambda} M(\mu)^2 d\mu \right) \mathbf{a}_\lambda^k + iM(\mu)k\mathbf{b}_k^\lambda.$$

This formula is a compatibility formula between the two numbers \mathbf{a}_λ^k and \mathbf{b}_k^λ . It appears after inspection that the term between parentheses may vanish as well. That's why we consider the particular solution of (2.17)

$$\mathbf{a}_\lambda^k = -iM(\lambda)k$$

and

$$\mathbf{b}_k^\lambda = k^2 + P.V. \int_{\mathbb{R}} \frac{\mu}{\mu - \lambda} M(\mu)^2 d\mu.$$

We can also completely determine \mathbf{c}_k^λ with (2.16): $\mathbf{c}_k^\lambda(\mu) = -\frac{i\alpha}{k}\psi_1(\mu)\alpha_k^\lambda = -\mu M(\mu)^2 M(\lambda)$. Finally the eigenvector V_λ^k may be written as

$$(2.18) \quad V_\lambda^k = -iM(\lambda)ke_1 + \left(k^2 + P.V. \int_{\mathbb{R}} \frac{\mu}{\mu - \lambda} M(\mu)^2 d\mu \right) U_\lambda - M(\lambda)P.V. \int_{\mathbb{R}} \frac{\mu}{\mu - \lambda} U_\mu M(\mu) d\mu.$$

For non zero k , the two last contributions cannot vanish at the same time. Therefore $V_\lambda^k \notin X$ and is indeed in the continuous spectrum.

Since

$$\begin{aligned} P.V. \int_{\mathbb{R}} \frac{\mu}{\mu - \lambda} U_\mu M(\mu) d\mu &= \int_{\mathbb{R}} U_\mu M(\mu) d\mu + \lambda P.V. \int_{\mathbb{R}} \frac{1}{\mu - \lambda} U_\mu M(\mu) d\mu \\ &= \alpha e_2 + \lambda P.V. \int_{\mathbb{R}} \frac{1}{\mu - \lambda} U_\mu M(\mu) d\mu \end{aligned}$$

using the orthogonality of the Hermite functions, it is possible to rewrite the eigenvector in a slightly different form

$$V_\lambda^k = -iM(\lambda)ke_1 - \alpha M(\lambda)e_2 + k^2U_\lambda + P.V. \int_{\mathbb{R}} \frac{\mu M(\mu)U_\lambda - \lambda M(\lambda)U_\mu}{\mu - \lambda} M(\mu)d\mu.$$

PROPOSITION 7 (Spectrum of A_k for $k \in \mathbb{R}^*$). *The spectrum of A_k , $k \neq 0$, is $S(A_k) = S_d(A_k) \cup S_c(A_k)$ where $S_d(A_k) = \{0\}$: the eigenvector is v_0^k (2.12). $S_c(A_k) = \mathbb{R}$: the generalized eigenvector associated to the eigenvalue λk is given in (2.18).*

This spectrum is complete.

Proof. The completeness that we desire to prove [10] is: if $U \in l^2$ is such that $(U, v_0^k) = 0$ and $(U, V_\lambda^k) = 0$ for all λ , then $U = 0$. To do so, let us use the notation that $U = (a_0, a_1, \dots)$ and define

$$g(\lambda) = \sum_{p \geq 1} a_p V_\lambda^k|_p$$

together with

$$f(\lambda) = \sum_{p \geq 1} a_p U_\lambda|_p.$$

The assumption $(U, v_0^k) = 0$ writes

$$a_0k + i\alpha a_1 = 0.$$

The assumption $(U, V_\lambda^k) = 0$ with V_λ^k given by (2.18) shows that

$$g(\lambda) = \left(k^2 + P.V. \int_{\mathbb{R}} \frac{\mu}{\mu - \lambda} M(\mu)^2 d\mu \right) f(\lambda) - M(\lambda)P.V. \int_{\mathbb{R}} \frac{\mu}{\mu - \lambda} f(\mu)M(\mu)d\mu + M(\lambda)ika_0.$$

Let us define the function

$$q(\lambda) := P.V. \int_{\mathbb{R}} \frac{\mu}{\mu - \lambda} M(\mu)^2 d\mu.$$

Simplifications yield

$$(k^2 + q(\lambda)) f(\lambda) - M(\lambda)P.V. \int_{\mathbb{R}} \frac{\mu}{\mu - \lambda} f(\mu)M(\mu)d\mu + \alpha M(\lambda)a_1 = 0.$$

By definition $\alpha a_1 = \int_{\mathbb{R}} f(\mu)M(\mu)d\mu$. So one obtains the equation

$$\begin{aligned} (k^2 + q(\lambda)) f(\lambda) - M(\lambda)P.V. \int_{\mathbb{R}} \frac{\mu}{\mu - \lambda} f(\mu)M(\mu)d\mu \\ + M(\lambda)P.V. \int_{\mathbb{R}} f(\mu)M(\mu)d\mu = 0, \end{aligned}$$

that is after simplification

$$(2.19) \quad (k^2 + q(\lambda)) f(\lambda) - \lambda M(\lambda) \int_{\mathbb{R}} \frac{1}{\mu - \lambda} f(\mu) M(\mu) d\mu = 0.$$

After splitting between real and imaginary parts, we can as well assume that f is a real quantity. Therefore multiplication by f and integration yield

$$(2.20) \quad \int_{\mathbb{R}} (k^2 + q(\lambda)) f(\lambda)^2 d\lambda - \int_{\mathbb{R}} f(\lambda) \lambda M(\lambda) P.V. \int_{\mathbb{R}} \frac{1}{\mu - \lambda} f(\mu) M(\mu) d\mu = 0.$$

- Let us first show that $q \leq \alpha^2$. By definition

$$\begin{aligned} q(v) &= P.V. \int_{\mathbb{R}} \frac{w}{v - w} M(w)^2 dw \\ &= -P.V. \int_{\mathbb{R}} \frac{1}{v - w} G'(w) dw = -\frac{d}{dv} \int_{\mathbb{R}} \frac{1}{v - w} G(w) dw = c'(v) \end{aligned}$$

where $c(v) = \int_{\mathbb{R}} \frac{1}{w - v} M(w)^2 dw$. The function c satisfies the equation

$$vc(v) + c'(v) = -P.V. \int_{\mathbb{R}} \frac{v - w}{v - w} M(w)^2 dw = -\alpha^2.$$

Since $c(0) = 0$, a direct integration shows that $c(v) = -\alpha^2 e^{-\frac{v^2}{2}} \int_0^v e^{\frac{s^2}{2}} ds$ and so

$$q(v) = -c'(v) = -\alpha^2 e^{-\frac{v^2}{2}} v \int_0^v e^{\frac{s^2}{2}} ds + \alpha^2.$$

It yields the upper bound $q(v) \leq \alpha^2$ for all v .

- We now show the lower bound $-\alpha^2 + \beta^2 \leq q$ for some $\beta > 0$. Assume that $v > 0$. Then

$$0 \leq v \int_0^v e^{\frac{s^2}{2}} ds \leq v \int_0^v e^{\frac{vs}{2}} ds = 2 \left(e^{\frac{v^2}{2}} - 1 \right).$$

In this case one gets the lower bound

$$\alpha^2 + q(v) \geq 2\alpha^2 - 2\alpha^2 \left(1 - e^{-\frac{v^2}{2}} \right) > 0 \quad \forall v \in \mathbb{R}.$$

But q is continuous and tends to zero at infinity. It shows the lower bound for $\beta > 0$

$$(2.21) \quad \alpha^2 + q(v) \geq \beta^2.$$

- Properties of the Hilbert transform yields that

$$\begin{aligned} A &= \int_{\mathbb{R}} f(\lambda) \lambda M(\lambda) \left(P.V. \int_{\mathbb{R}} \frac{1}{\mu - \lambda} f(\mu) M(\mu) d\mu \right) d\lambda \\ &= - \int_{\mathbb{R}} \left(P.V. \int_{\mathbb{R}} \frac{\mu}{\mu - \lambda} f(\mu) M(\mu)^2 d\mu \right) (f(\lambda) M(\lambda)) d\lambda \end{aligned}$$

Therefore A is the half sum of these two expressions

$$\begin{aligned} A &= -\frac{1}{2} \int_{\mathbb{R}} f(\lambda) M(\lambda) \left(P.V. \int_{\mathbb{R}} \frac{\mu - \lambda}{\mu - \lambda} f(\mu) M(\lambda) d\mu \right) d\lambda \\ &= -\frac{1}{2} \left(\int_{\mathbb{R}} f(\lambda) M(\lambda) d\lambda \right)^2. \end{aligned}$$

- So (2.20) yields

(2.22)

$$\int_{\mathbb{R}} \left(k^2 + P.V. \int_{\mathbb{R}} \frac{\mu}{\mu - \lambda} M(\mu)^2 d\mu \right) f(\lambda)^2 d\lambda + \frac{1}{2} \left(\int_{\mathbb{R}} f(\lambda) M(\lambda) d\lambda \right)^2 = 0.$$

Since the function q is bounded, the kernel $k^2 + q(\lambda)$ is positive for k^2 large enough. In this case it shows that f vanishes, which ultimately shows that $g = 0$ and $a_0 = 0$. If k^2 is small but positive, the result follows from the analysis of the operator L in the next section, see proposition 15. The completeness of the Hermite polynomials yields that $U = 0$.

□

Let us assume $k \neq 0$ and the Gauss compatibility condition (1.7) is satisfied. From spectral representation theorems [10] for non bounded hermitian operators, there is in theory a weight $\lambda \mapsto w(\lambda)$ such that the representation formula holds

$$(2.23) \quad U = \frac{1}{\alpha^2 + k^2} (U, v_0^k) v_0^k + \int_{\mathbb{R}} (U, V_\lambda^k) V_\lambda^k w(\lambda) d\lambda, \quad U \in l^2.$$

Notice that the contribution of the discrete eigenvector is normalized in this formula since $\|v_0^k\|^2 = \alpha^2 + k^2$. One also has the energy relation

$$(2.24) \quad \|U\|^2 = \frac{1}{\alpha^2 + k^2} |(U, v_0^k)|^2 + \int_{\mathbb{R}} |(U, V_\lambda^k)|^2 w(\lambda) d\lambda.$$

Even we do not know explicitly the weight w , let us analyze some consequences of these formulas. The action of the exponential (2.8) writes

$$(2.25) \quad e^{-iA_k t} W = (W, v_0^k) u_0^k + \int_{\mathbb{R}} e^{-i\mu t} (W, V_\mu^k) V_\mu^k w(\mu) d\mu.$$

The initial Gauss compatibility condition $\partial_x E = -\int_{\mathbb{R}} g dv = -\alpha \int_{\mathbb{R}} w \psi_0(v) dv$ is equivalent to $ik(W, e_1) + \alpha(W, e_2) = 0$ that is the orthogonality

$$(W, v_0^k) = 0.$$

So the corresponding term vanishes in (2.25). One sees that

$$e^{-iA_k t} W = \int_{\mathbb{R}} e^{-i\mu t} (W, V_\mu^k) V_\mu^k w(\mu) d\mu.$$

The linear Landau damping is a consequence. Let $Z \in X$. Then as a consequence

$$(Z, e^{-iA_k t} W) = \int_{\mathbb{R}} e^{-i\mu t} (Z, V_\mu^k) (W, V_\mu^k) w(\mu) d\mu$$

tends to zero as $t \rightarrow 0$, provided the function $\mu \mapsto k(\mu) = (Z, V_\mu^k) (W, V_\mu^k) w(\mu)$ is smooth enough. We see that phase mixing if the reason of linear Landau damping, which is a well known fact [4, 17, 13]. However we will not pursue in this direction since a more direct and quantitative technique will now be considered.

3. The operator L . To be more general we partially relax the Maxwellian hypothesis. We consider a solution (g, E) of the linear Vlasov-Ampère equation (1.6) where G is a bump. That is we assume there exists a smooth positive function $v \mapsto M(v) > 0$ such that

$$(3.1) \quad G'(v) = -vM^2(v).$$

It is still possible to renormalize $u = \frac{g}{M}$ so that

$$(3.2) \quad \begin{cases} \partial_t u + v \partial_x u &= -vME, & t > 0, & (x, v) \in I \times \mathbb{R}, \\ \partial_t E &= \int_{\mathbb{R}} uvMdv, & t > 0, & x \in I, \\ \partial_x E &= -\int_{\mathbb{R}} uMdv, & t > 0, & x \in I, \end{cases}$$

still holds. It is immediately possible to determine the family of polynomials which are orthonormal with respect to the weight $G = M^2$ and to determine generalized Hermite functions ψ_n^M . These functions are a natural generalization of the Hermite functions (2.2). The difference with the previous material is mainly the coefficients of the recursion formula (2.3) which implies a modification of the matrices A , B and A_k . But up this modification, it is possible to generalize the previous analysis and to determine the eigenvectors and the spectrum of A_k even for a non Maxwellian profile M .

That is why we consider that $M > 0$ is a positive weight which satisfies three main properties.

1. $M(v)$ decreases sufficiently fast to zero for $|v| \rightarrow 0$ so that (3.1) holds and the family of generalized Hermite functions is an Hilbert basis of $L^2(\mathbb{R})$

$$(3.3) \quad \forall u \in L^2(\mathbb{R}), \exists (\alpha_n) \in l^2 \text{ such that } u = \sum_n \alpha_n \psi_n^M,$$

$$\text{and } \|u\|_{L^2(\mathbb{R})}^2 = \sum_{n \in \mathbb{N}} |\alpha_n|^2.$$

2. The mass of M^2 is arbitrary. So we redefine the constant α

$$(3.4) \quad \alpha_M^2 = \int_{\mathbb{R}} M(v)^2 dv.$$

3. We finally assume regularity of the weight as follows

$$(3.5) \quad M \in H^n(\mathbb{R}) \quad \forall n \in \mathbb{N}^*.$$

Let us define the integro-differential operator which corresponds to the Fourier transform of the operator seen in (2.19). L is the sum of a Poisson operator in the x variable plus an integral operator in the v variable.

DEFINITION 8. *The operator integro-differential L is defined by*

$$(3.6) \quad Lu(x, v) = (-\partial_{xx} + q(v)) u(x, v) - vM(v)P.V. \int_{\mathbb{R}} \frac{1}{w-v} u(x, w)M(w)dw.$$

This operator has quite strong properties.

PROPOSITION 9. *Take u that satisfies*

$$\partial_t u + v \partial_x u + vME = 0$$

together with the Gauss law $\partial_x E(t, x) + \int_{\mathbb{R}} u(t, x, v) M(v) dv = 0$. Then $h = Lu$ satisfies the transport equation in free space

$$(3.7) \quad \partial_t h + v \partial_x h = 0.$$

Proof.

Let us start from

$$\partial_t u + v \partial_x u + v M(v) E = 0.$$

Since $\partial_t L = L \partial_t$ one has

$$\partial_t h + L(v \partial_x u) + L(v M(v) E) = 0.$$

The second term is

$$\begin{aligned} L(v \partial_x u) &= v \partial_x ((-\partial_{xx} + q(v))u) - v M(v) P.V. \int_{\mathbb{R}} \frac{1}{w-v} w \partial_x u(x, w) M(w) dw \\ &= v \partial_x Lu - v M(v) \int_{\mathbb{R}} \partial_x u(x, w) M(w) dw = v \partial_x Lu - v M(v) \partial_x \int_{\mathbb{R}} u(x, w) M(w) dw \end{aligned}$$

where we have used that $\frac{w}{w-v} = \frac{v}{w-v} + 1$. The last term is

$$\begin{aligned} L(v M(v) E) &= v M(v) (-\partial_{xx} + q(v)) E - E v M(v) P.V. \int_{\mathbb{R}} \frac{w}{w-v} M(w)^2 dw \\ &= v M(v) (-\partial_{xx} + q(v)) E - E v M(v) q(v) = -v M(v) \partial_{xx} E. \end{aligned}$$

The Gauss law $\partial_x E + \int_{\mathbb{R}} u(w) M(w) dw = 0$ implies cancellations in the sum of these two terms which simplifies into

$$L(v \partial_x u) + L(v M(v) E) = v \partial_x Lu = v \partial_x h.$$

The proof is ended. \square

For convenience we define \mathcal{I} the part of L which acts on the velocity variable

$$\mathcal{I}f(v) = q(v)f(v) - v M(v) P.V. \int_{\mathbb{R}} \frac{1}{w-v} f(w) M(w) dw.$$

PROPOSITION 10. *The operator \mathcal{I} is continuous from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$.*

Proof. First the function q is bounded so $f \mapsto qf$ is continuous. Second the Hilbert transform is an isometry, see [16] in $L^2(\mathbb{R})$. It implies that

$$\begin{aligned} &\left\| P.V. \int_{\mathbb{R}} \frac{1}{w-v} f(w) M(w) dw \right\|_{L^2(\mathbb{R})} \\ &= \|f(w) M(w)\|_{L^2(\mathbb{R})} \leq \|M\|_{L^\infty(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}. \end{aligned}$$

The proof is ended. \square

Next we study the structure of \mathcal{I} . Two preliminary results in this direction are the following.

PROPOSITION 11. *One has $\mathcal{I}M = \alpha_M^2 M$.*

Proof. By definition

$$\mathcal{I}M(v) = q(v)M(v) - M(v)P.V. \int \frac{v}{w-v} M(w)^2 dw.$$

But

$$P.V. \int \frac{v}{w-v} M(w)^2 dw = P.V. \int \frac{w}{w-v} M(w)^2 dw - \int M(w)^2 dw = q(v) - \alpha_M^2.$$

It ends the proof. \square

PROPOSITION 12. *One has $\mathcal{I}(vM) = 0$.*

Proof. Indeed

$$\mathcal{I}(vM) = q(v)vM(v) - vM(v)P.V. \int \frac{w}{w-v} M(w)^2 dw = 0$$

by definition of q . \square

By linearity we obtain that $\mathcal{I}\psi_0^M = \alpha_M^2 \psi_0^M$ and $\mathcal{I}\psi_1^M = 0$. The general case is as follows.

PROPOSITION 13. *Let $n \geq 2$. Then $\mathcal{I}(\psi_n^M) \in \text{Span}_{p \leq n-1} \{\psi_p^M\}$.*

Proof. We use the fact that $\psi_n^M(v) = I_n^M(v)M(v)$ where $I_n^M(v)$ is a polynomial of degree n . One has the formula

$$\begin{aligned} \mathcal{I}\psi_n^M &= M(v) \int_{\mathbb{R}} \frac{wI_n^M(v) - vI_n^M(w)}{w-v} M(w)^2 dw \\ &= M(v) \int_{\mathbb{R}} \frac{I_n^M(v) - I_n^M(w)}{w-v} M(w)^2 dw - M(v) \underbrace{\int_{\mathbb{R}} I_n^M(w) M(w)^2 dw}_{=0} \end{aligned}$$

where the last vanishes since $\int_{\mathbb{R}} I_n^M(w) I_0^M(w) M(w)^2 dw = 0$. The function $\frac{I_n(v) - I_n(w)}{w-v}$ is a polynomial of partial degree at most $n-1$ with respect to v or with respect to w . So $\int_{\mathbb{R}} v \frac{I_n(v) - I_n(w)}{w-v} M(w)^2 dw \in \text{Span}_{p \leq n-1} \{I_p\}$. The result follows. \square

PROPOSITION 14. *The operator \mathcal{I} is Hilbert-Schmidt.*

Proof. We split $\mathcal{I} = \mathcal{I}_1 - \mathcal{I}_2$. First $\mathcal{I}_1 f = q(v)f(v)$ is Hilbert-Schmidt since

$$\sum_n \|\mathcal{I}_1 \psi_n^M\|_{L^2(\mathbb{R})}^2 = \sum_n \|q(v)\psi_n^M\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} q(v)^2 dv = \|q\|_{L^2(\mathbb{R})}^2.$$

By definition $q(v) = P.V. \int_{\mathbb{R}} \frac{1}{w-v} wM(w)^2 dw$. The isometry of the Hilbert transform yields

$$\|q\|_{L^2(\mathbb{R})}^2 = \|wM(v)^2\|_{L^2(\mathbb{R})}^2 < \infty.$$

The second part is

$$\mathcal{I}_2 f(v) = vM(v)P.V. \int_{\mathbb{R}} \frac{1}{w-v} f(w)M(w)dw.$$

Let us define $C = \|vM(v)\|_{L^\infty(\mathbb{R})}$ so that

$$\|\mathcal{I}_2 \psi_n\|_{L^2(\mathbb{R})} \leq C \left\| P.V. \int_{\mathbb{R}} \frac{1}{w-v} \psi_n(w) M(w) dw \right\|_{L^2(\mathbb{R})} = C \|\psi_n M(v)\|_{L^2(\mathbb{R})}.$$

Therefore

$$\sum_n \|\mathcal{I}_2 \psi_n\|_{L^2(\mathbb{R})}^2 \leq C^2 \|\psi_n M(w)\|_{L^2(\mathbb{R})}^2 = C^2 \int_{\mathbb{R}} M(w)^2 dw < \infty.$$

Since the difference of two Hilbert-Schmidt operators is Hilbert-Schmidt, the proof is ended. \square

Consider

$$L_k = k^2 I_d + \mathcal{I} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$$

PROPOSITION 15. *Whatever $k \neq 0$, L_k and L_k^* are invertible in $\mathcal{L}(L^2(\mathbb{R}))$ with continuous inverse. Moreover there exists a constant $K > 0$ such that $\|L_k^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}))} = \|L_k^{-*}\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \frac{C}{k^2}$ for all $k \in \mathbb{Z}^*$.*

Proof. By propositions 11 to 13, \mathcal{I}^* can be represented as a infinite lower triangular operator in the basis (ψ_n^M) with a non negative diagonal

$$\mathcal{I}^* = \begin{pmatrix} \alpha^2 & 0 & 0 & 0 & \dots \\ \times & 0 & 0 & 0 & \dots \\ \times & \times & 0 & 0 & \dots \\ \times & \times & \times & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

So L_k^* has the same structure but with a positive diagonal

$$L_k^* = \begin{pmatrix} k^2 + \alpha^2 & 0 & 0 & 0 & \dots \\ \times & k^2 & 0 & 0 & \dots \\ \times & \times & k^2 & 0 & \dots \\ \times & \times & \times & k^2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

So L_k^* is into. Since $L_k^* = k^2 \left(I_d + \frac{\mathcal{I}^*}{k^2} \right)$ where $\frac{\mathcal{I}^*}{k^2}$ is a compact operator (since Hilbert-Schmidt), the operator L_k^* is invertible with continuous inverse. By duality L_k is also invertible.

Assume $k^2 > 2\|\mathcal{I}\|_{\mathcal{L}(L^2(\mathbb{R}))}$: the Neumann series is $L_k^{-1} = \frac{1}{k^2} \sum_{n \geq 0} \left(-\frac{\mathcal{I}}{k^2} \right)^n$. So

$$\|L_k^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \frac{1}{k^2} \sum_{n \geq 0} \frac{1}{2^n} = \frac{2}{k^2}.$$

Next we define

$$K = \max_{k \in \mathbb{Z}^*} (k^2 \|L_k^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}))}) = \max \left(\max_{k \neq 0; k^2 \leq 2\|\mathcal{I}\|_{\mathcal{L}(L^2(\mathbb{R}))}} k^2 \|L_k^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}))}, 2 \right) < \infty.$$

Same bound holds for the inverse of L_k^* . The proof is ended. \square

PROPOSITION 16. *Let $n \in \mathbb{N}^*$. There exists a constant $C_n > 0$ such that*

$$\|L_k^{-1}\|_{\mathcal{L}(H^n(\mathbb{R}))} = \|L_k^{-*}\|_{\mathcal{L}(H^n(\mathbb{R}))} \leq \frac{C_n}{k^2}, \quad k \in \mathbb{Z}^*.$$

Proof. • We first detail the proof for $n = 1$ and the direct operator. Let us consider the unique solution of the problem $L_k f = g$ where $f, g \in L^2(\mathbb{R})$. Assume $g \in H^1(\mathbb{R})$. Then one can derive the equation $L_k f = g$ to get

$$L_k \partial_v f(v) = \partial_v g(v) + h(v)$$

where $h = -(\partial_v L_k)f$, that is

$$\begin{aligned} h(v) &= -q'(v)f(v) + \frac{d}{dv}(vM(v)) P.V. \int_{\mathbb{R}} \frac{1}{w-v} f(w)M(w)dw \\ &\quad + vM(v) P.V. \int_{\mathbb{R}} \frac{1}{w-v} f(w) \frac{d}{dw} M(w)dw. \end{aligned}$$

By definition $\|h\|_{L^2(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})} \leq c\|g\|_{L^2(\mathbb{R})}$ for some constants $C, c > 0$. Using the result of the previous proposition, one gets

$$\|\partial_v f\|_{L^2(\mathbb{R})} \leq \frac{C}{k^2} (\|\partial_v g\|_{L^2(\mathbb{R})} + \|g\|_{L^2(\mathbb{R})}) \leq \frac{\tilde{C}}{k^2} \|g\|_{H^1(\mathbb{R})}.$$

The result is shown for $n = 1$.

• The proof for all $n > 1$ can be done by using the Leibniz formula for the expansion of the derivatives

$$\frac{d^n}{dv^n}(L_k f) = \sum_{p=0}^n \frac{n!}{p!(n-p)!} \frac{d^p}{dv^p} L_k \frac{d^{n-p}}{dv^{n-p}} f,$$

and the fact that $\frac{d^p}{dv^p} L_k$ is a bounded operator in $\mathcal{L}(L^2(\mathbb{R}))$ for all p . It yields

$$L_k \frac{d^n}{dv^n} f = \frac{d^n}{dv^n} g - \sum_{p=1}^n \frac{n!}{p!(n-p)!} \frac{d^p}{dv^p} L_k \frac{d^{n-p}}{dv^{n-p}} f$$

from which the proof is completed after recursion.

• By duality, same bounds hold for L_k^{-*} . \square

3.1. Linear Landau damping. We apply the previous material to the analysis of the linear Landau damping phenomenon which expresses the fact that the electric field is damped and tends to zero as time tends to infinity. Our formulation is slightly different from the usual one. We start from the electric potential ϕ defined by

$$\partial_x \phi = -E.$$

This makes sense since $\int_I E dx = 0$.

THEOREM 17 (Linear Landau Damping). *Consider a solution (u, E) of the linearized equations (3.2). Assume the initial condition is $u_0 \in L^2(I : H^n(\mathbb{R}))$. There exists a constant $C_n > 0$ such that*

$$(3.8) \quad \|\phi(t)\|_{W^{1,\infty}(I)} + \|\partial_t \phi(t)\|_{L^\infty(I)} \leq \frac{C_n}{\max(1, t^n)} \|u_0\|_{L^2(I : H^n(\mathbb{R}))}.$$

Proof. From the Gauss law $\partial_x E = - \int_{\mathbb{R}} u(x, v) M(v) dv$ and the fact that

$$\int_I \int_{\mathbb{R}} u(x, v) M(v) dv dx = 0,$$

there exists a continuous function $a_x : I \rightarrow I$ such that

$$(3.9) \quad E(t, x) = \int_I \int_{\mathbb{R}} u(t, y, v) a_x(y) M(v) dx dv.$$

Next one has that

$$\partial_x \partial_t \phi = \partial_t E = \int_{\mathbb{R}} v u(v) M(v) dv.$$

Therefore $\partial_t \phi$ can be represented by a formula similar to (3.9)

$$(3.10) \quad \partial_t \phi(t, x) = - \int_I \int_{\mathbb{R}} u(t, y, v) a_x(y) v M(v) dx dv.$$

The bound

$$\|\phi(t)\|_{W^{1,\infty}(I)} + \|\partial_t \phi(t)\|_{L^\infty(I)} \leq C \|u_0\|_{L^2(I; L^2(\mathbb{R}))}$$

is evident from these expressions. So the point is to get the $\frac{1}{t^n}$ behavior.

- The function $a_x(y)$ is defined up to a constant. A easy calculation shows that

$$a_x(y) = \begin{cases} \frac{-y + x - \pi}{2\pi} & 0 < y < x \\ \frac{-y + x + \pi}{2\pi} & x < y < 2\pi \end{cases}$$

is convenient. This function has vanishing mean in the sense that $\int_I a_x(y) dy = 0$ and $\int_I a_x(y) dx = 0$. So there exists $(a_k(x))_{k \in \mathbb{Z}^*}$ such that

$$a_x(y) = \frac{1}{2\pi} \sum_{k \neq 0} a_k(x) e^{-iky}.$$

Precisely

$$a_k(x) = \int_I a_x(y) e^{iky} dy = \frac{1}{ik} e^{ikx} \quad k \neq 0.$$

Note that the normalization of the Fourier transform is here different from (2.5): this is useful to obtain simpler intermediate notations. One has the representation formula

$$a_x(y) M(v) = \frac{1}{2\pi} \sum_{k \neq 0} a_k(x) e^{-iky} M(v).$$

- On the other hand, the boundedness of L_k^{-1} provided by propositions 15-16, yields similar estimates for $(L_k^*)^{-1}$. So there exists $b_k \in L^2(\mathbb{R})$ such that

$$L_k^* b_k(v) = M(v), \quad k \neq 0.$$

Since the function $v \mapsto M(v)$ is by hypothesis in all $H^n(\mathbb{R})$, one has for every $n \in \mathbb{N}^*$

$$\|b_k\|_{H^n(\mathbb{R})} \leq \frac{C_n}{|k|^2} \quad k \neq 0.$$

Let us define the function

$$(3.11) \quad b_x(y, v) = \sum_{k \neq 0} a_k(x) e^{-iky} b_k(v)$$

Equipped with these notations, one can rewrite (3.9) as

$$(3.12) \quad E(t, x) = \langle u, a_x M \rangle = \langle u, L^* b_x \rangle = \langle Lu, b_x \rangle = \langle h, b_x \rangle.$$

It yields the integral formula

$$(3.13) \quad E(t, x) = \int_I \int_{\mathbb{R}} h(t, y, v) b_x(y, v) dx dv = \int_I \int_{\mathbb{R}} h_0(y - vt, v) b_x(y, v) dx dv.$$

• Let us now bound the integral by means of Fourier decomposition. The initial value is $h_0 = Lu_0$ where

$$(3.14) \quad u_0(x, v) = \sum_k e^{ikx} m_k(v).$$

So one gets the Fourier representation

$$(3.15) \quad h_0(x, v) = \sum_k e^{ikx} c_k(v) \quad \text{with } c_k(v) = L_k m_k(v).$$

The boundedness of $u_0 \in L^2(I : H^n(\mathbb{R}))$ is characterized by

$$\|u_0\|_{L^2(I : H^n(\mathbb{R}))}^2 = \sum_k \|m_k\|_{H^n(\mathbb{R})}^2 < \infty$$

which yields

$$\|h_0\|_{L^2(I : H^n(\mathbb{R}))}^2 \leq \sum_k (k^2 + D_n) \|m_k\|_{H^n(\mathbb{R})}^2$$

where $D_n = \|\mathcal{I}\|_{\mathcal{L}(H^n(\mathbb{R}))} > 0$ is a constant independent of k . The electric field is obtained by inserting (3.11) and (3.15) into (3.13)

$$E(t, x) = \sum_{k \neq 0} \int_{\mathbb{R}} e^{-ikvt} c_k(v) a_k(x) b_{-k}(v) dv.$$

Integration by parts with respect to v shows that

$$\int_{\mathbb{R}} e^{-ikvt} c_k(v) b_k(v) dv = \frac{1}{(-ikt)^n} \int_{\mathbb{R}} e^{-ikvt} \frac{d^n}{dv^n} (c_k b_k)(v) dv, \quad k \neq 0.$$

It yields

$$\left| \int_{\mathbb{R}} e^{-ikvt} c_k(v) b_k(v) dv \right| \leq \frac{E_n}{|k|^n t^n} \|c_k\|_{H^n(\mathbb{R})} \|b_k\|_{H^n(\mathbb{R})}$$

$$\leq \frac{E_n}{|k|^n t^n} ((k^2 + D_n) \|m_k\|_{H^n(\mathbb{R})}) \left(\frac{C_n}{k^2} \right) \leq \frac{F_n}{|k|^n t^n} \|m_k\|_{H^n(\mathbb{R})}.$$

So

$$|E(t, x)| \leq \sum_{k \neq 0} \frac{F_n}{|k|^n t^n} \|m_k\|_{H^n(\mathbb{R})} |a_k(x)| \leq \frac{G_n}{|k|^{n+1} t^n} \|m_k\|_{H^n(\mathbb{R})}.$$

A standard Cauchy-Schwarz inequality yields

$$(3.16) \quad |E(x, t)| \leq \frac{G_n}{t^n} \left(\sum_{k \neq 0} \|m_k\|_{H^n(\mathbb{R})}^2 \right)^{\frac{1}{2}} \left(\sum_{k \neq 0} \frac{1}{k^{2n+2}} \right)^{\frac{1}{2}} \leq \frac{H_n}{t^n} \|u_0\|_{L^2(I; H^n(\mathbb{R}))}.$$

It shows the result for $\partial_x \phi = -E$, and also for ϕ by integration in the x direction. From (3.10) the same inequality is obtained for $\partial_t \phi$. The proof is ended.

□

4. Influence of the Debye length. We now generalize the integro-differential operator L to the non linear equation (1.9). Our objective is to exploit the explicit nature of the operator L to analyze the sensitivity of the damping rate of the electric field with respect to the Debye length λ_D . To simplify the analysis, in particular the proof of proposition 19, we assume that $G(v) = \exp(-\frac{v^2}{2})$. That is we consider that

$$(4.1) \quad M(v) = \exp(-\frac{v^2}{4})$$

in this section.

The electronic density can be written as $n_e = \alpha^2 \exp \phi$ so that the linearization of (1.9) is based on the formal decomposition

$$f = G + g, \quad E = 0 + e, \quad n_e = \alpha^2(1 + \phi + O(\phi^2)).$$

It yields the linear problem

$$(4.2) \quad \begin{cases} \partial_t g + v \partial_x g + e \partial_v G = 0, & t > 0, \quad (x, v) \in I \times \mathbb{R}, \\ -\lambda_D^2 \partial_{xx} \phi + \alpha^2 \phi = \int_{\mathbb{R}} g dv, & t > 0, \quad x \in I, \\ e = -\partial_x \phi, & t > 0, \quad x \in I. \end{cases}$$

One has $G'(v) = -vM^2(v)$. Let us define $u = \frac{g}{M}$ which is solution of

$$(4.3) \quad \partial_t u + v \partial_x u - evM(v) = 0.$$

The convenient generalization of the integro-differential operator L is L^D

$$(4.4) \quad L^D u(x, v) = (-\lambda_D^2 \partial_{xx} + \alpha^2 + q(v)) u(x, v) - vM(v) P.V. \int_{\mathbb{R}} \frac{1}{w - v} u(x, w) M(w) dw.$$

The companion definition for mode $k \in \mathbb{Z}$ is L_k^D

$$L_k^D u(v) = (\lambda_D^2 k^2 + \alpha_M^2 + q(v)) u(v) - vM(v) P.V. \int_{\mathbb{R}} \frac{1}{w - v} u(w) M(w) dw.$$

PROPOSITION 18. *Let u be a solution of (4.3) with the electric field given by (4.2). The function $h^D = L^D u$ is solution of the transport equation*

$$\partial_t h^D + v \partial_x h^D = 0.$$

Proof. The structure of the proof is the same as the one of proposition 9. First one has $L \partial_t u = \partial_t h^D$. Second

$$\begin{aligned} L^D(v \partial_x u) &= (-\lambda_D^2 \partial_{xx} + \alpha_M^2 + q(v)) v \partial_x u(x, v) \\ &\quad - v M(v) P.V. \int_{\mathbb{R}} \frac{1}{w-v} w \partial_x u(x, w) M(w) dw \\ &= v \partial_x (-\lambda_D^2 \partial_{xx} + \alpha_M^2 + q(v)) u(x, v) \\ &\quad - v M(v) P.V. \int_{\mathbb{R}} \left(\frac{w-v}{w-v} + \frac{v}{w-v} \right) \partial_x u(x, w) M(w) dw \\ &= v \partial_x (-\lambda_D^2 \partial_{xx} + \alpha_M^2 + q(v)) u(x, v) \\ &\quad + v \partial_x \left(-v M(v) P.V. \int_{\mathbb{R}} \left(\frac{1}{w-v} \right) u(x, w) M(w) dw \right) \\ &\quad - v M(v) \partial_x \int_{\mathbb{R}} u(x, w) M(w) dw = v \partial_x L^D u - v M(v) \partial_x \int_{\mathbb{R}} u(x, w) M(w) dw. \end{aligned}$$

And third

$$\begin{aligned} L^D(evM(v)) &= (-\lambda_D^2 \partial_{xx} + \alpha_M^2 + q(v)) evM(v) \\ &\quad - v M(v) P.V. \int_{\mathbb{R}} \frac{1}{w-v} (ewM(w)) M(w) dw \\ &= v M(v) (-\lambda_D^2 \partial_{xx} + \alpha_M^2) e = -v M(v) \partial_x (-\lambda_D^2 \partial_{xx} + \alpha_M^2) \phi \end{aligned}$$

due to the last equation of (4.2). One gets after summation

$$0 = \partial_t h^D + v \partial_x h^D + v M(v) \partial_x \left(\int_{\mathbb{R}} u(x, w) M(w) dw - (-\lambda_D^2 \partial_{xx} + \alpha_M^2) \phi \right),$$

where the term between parenthesis vanishes in view of the second equation of (4.2). The proof is ended. \square

It is convenient to define the velocity part of the operator L^D

$$\mathcal{I}^D u(v) = (\alpha_M^2 + q(v)) u(v) - v M(v) P.V. \int_{\mathbb{R}} n \frac{1}{w-v} u(w) M(w) dw.$$

PROPOSITION 19. *The operator \mathcal{I}^D is coercive in $L^2(\mathbb{R})$.*

REMARK 20. *This is not the case for \mathcal{I} which was proved only to be Hilbert-Schmidt.*

Proof. The calculations are based on the ones that give (2.22). One has

$$(\mathcal{I}^D u, u)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} (\alpha_M^2 + q(v)) u(v)^2 dv + \frac{1}{2} \left(\int_{\mathbb{R}} u(v) M(v) dv \right)^2.$$

The result follows from the double inequality $-\alpha^2 + \beta^2 \leq q \leq \alpha$ that was proved in (2.21) for the Maxwellian profile. \square

The proposition 16 is then easily generalized. A consequence is that L_k^D has continuous inverse in $H^n(\mathbb{R})$ with the bound

$$\| (L_k^D)^{-1} \|_{\mathcal{L}(H^n(\mathbb{R}))} \leq \frac{C_n}{\lambda_D^2 k^2 + \beta^2}, \quad \beta > 0.$$

Next step is the writing of an integral formula for the electric field. From $-\lambda_D^2 \partial_{xx} \phi + \alpha^2 \phi = \int_{\mathbb{R}} g dv = \int_{\mathbb{R}} u M dv$, one can write the Fourier representation formula

$$\phi(x) = \sum_{k \neq 0} \frac{1}{2\pi(\lambda_D^2 k^2 + \alpha^2)} \left(\int_I \int_{\mathbb{R}} e^{-iky} u(y, v) M(v) dy dv \right) e^{ikx},$$

also rewritten in a more compact form as $\phi(x) = \int_I \int_{\mathbb{R}} u(y, v) K_x(y, v) dy dv$ with

$$K_x(y, v) = \sum_{k \neq 0} \frac{1}{2\pi(\lambda_D^2 k^2 + \alpha^2)} e^{ik(x-y)}.$$

The electric field is directly

$$e(x) = -\partial_x \phi = \int_I \int_{\mathbb{R}} u(y, v) \hat{K}_x(y, v) dy dv$$

with $\hat{K}_x = \partial_x K_x$

$$\hat{K}_x(y, v) = \sum_{k \neq 0} \frac{ik}{2\pi(\lambda_D^2 k^2 + \alpha^2)} e^{ik(x-y)} M(v).$$

One can write directly

$$e(t, x) = \langle Lu, (L^D)^{-*} K_x \rangle = \langle h, (L^D)^{-*} K_x \rangle$$

since \mathcal{I}^D is coercive in $L^2(\mathbb{R})$ which yields that L^D is invertible. We obtain in Fourier

$$e(t, x) = \int_I \int_{\mathbb{R}} \left(h_0(y - vt, v) \sum_{k \neq 0} \frac{ik}{2\pi(\lambda_D^2 k^2 + \alpha^2)} e^{ik(x-y)} (L_k^D)^{-*} M(v) \right) dy dv.$$

Using the expansion (3.15) of the initial data h_0 , one gets

$$e(t, x) = \sum_{k \neq 0} \frac{ik e^{ikx}}{2\pi(\lambda_D^2 k^2 + \alpha^2)} \int_{\mathbb{R}} e^{-ikvt} c_k(v) (L_k^D)^{-*} M(v) dv$$

Since $\|(L_k^D)^{-*}M(v)\|_{H^n(\mathbb{R})} \leq \frac{D_n}{\lambda_D^2 k^2 + \beta^2}$, one gets the bound after integration by parts in the v variable

$$|e(t, x)| \leq C \sum_{k \neq 0} \frac{|k|}{(\lambda_D^2 k^2 + \alpha^2)(\lambda_D^2 k^2 + \beta^2)(|k|^n t^n)} \|c_k\|_{H^n(\mathbb{R})}.$$

Here $c_k = L_k^D m_k$ and m_k is the Fourier decomposition (3.14) of the initial data u_0 . Let $\gamma_n^2 > 0$ be a measure of the norm of \mathcal{I}^D in $\mathcal{L}(H^n(\mathbb{R}))$. One can writes

$$\begin{aligned} \|e(t)\|_{L^\infty(I)} &\leq \frac{C}{t^n} \sum_{k \neq 0} \frac{|k|^{1-n} (\lambda_D^2 k^2 + \gamma_n^2)}{(\lambda_D^2 k^2 + \alpha^2)(\lambda_D^2 k^2 + \beta^2)} \|m_k\|_{H^n(\mathbb{R})} \\ &\leq \frac{\hat{C}_n}{t^n} \sum_{k \neq 0} \frac{|k|^{1-n}}{(\lambda_D^2 k^2 + \beta^2)} \|m_k\|_{H^n(\mathbb{R})}. \end{aligned}$$

Finally a Cauchy-Schwarz inequality yields

$$(4.5) \quad \|e(t)\|_{L^\infty(I)} \leq \frac{\hat{C}_n}{t^n} \|u_0\|_{L^2(I; H^n(\mathbb{R}))} \left(\sum_{k \neq 0} \frac{1}{(\lambda_D^2 k^2 + \beta^2)^2 |k|^{2n-2}} \right)^{\frac{1}{2}}.$$

If $\lambda > 0$ is considered as a fixed positive coefficient, this formula does not bring anything new with respect to (3.16). The interesting physical case when λ^D tends to zero is discussed below.

THEOREM 21. *Assume that $u_0 \in L^2(I; H^n(\mathbb{R}))$ with $n \geq 2$. Then the electric field tends to zero uniformly with respect to the Debye length with a rate at least t^{-n} . This is evident from (4.5).*

REMARK 22. *On the other hand if $n = 1$ and $\lambda_D = 0$ one cannot predict any damping rate from (4.5) since the series diverges.*

5. First non linear estimates. We desire to show that some parts of the symetrization method can be generalized to the non linear case. This can be considered as a first and elementary step in the understanding of the non linear Landau damping with the tools developed in this work. It also answers positively to a question addressed in [17] about the validity of quadratic estimates for non linear equations.

Keeping the non linearity in the Vlasov-Poisson system (1.3) yields

$$(5.1) \quad \begin{cases} \partial_t g + v \partial_x g + EvM^2 &= E \partial_v g, & t > 0, & (x, v) \in I \times \mathbb{R}, \\ \partial_t E - \int v g dv &= 0, & t > 0, & x \in I. \end{cases}$$

Rewriting as before with $u = \frac{g}{M}$ one gets

$$\begin{cases} \partial_t u + v \partial_x u + vME &= E \frac{1}{M} \partial_v g, & t > 0, & (x, v) \in I \times \mathbb{R}, \\ \partial_t E - \int uvM dv &= 0, & t > 0, & x \in I. \end{cases}$$

which is more conveniently rewritten as

$$(5.2) \quad \begin{cases} \partial_t u + v \partial_x u + vME &- E \partial_v u &= \frac{1}{2} v E u, & t > 0, & (x, v) \in I \times \mathbb{R}, \\ \partial_t E &- \int uvM dv &= 0, & t > 0, & x \in I. \end{cases}$$

Only the right hand side has influence on the L^2 energy balance.

The quadratic term $E\partial_v u$ is compatible with the quadratic estimates pursued in this work because this term vanishes after multiplication by u and integration. However it is not the case of the other term $\frac{1}{2}vEu$. Indeed a direct use of (5.2) yields

$$\frac{d}{dt} \left(\int_I \int_{\mathbb{R}} u^2 dx dv + \int_I E^2 dx \right) = \int_I \left(E \int_{\mathbb{R}} vu^2 dv \right) dx.$$

Since $\int_{\mathbb{R}} |v|u^2 dv$ cannot be controlled by $\int_{\mathbb{R}} u^2 dv$, it shows a defect in the control of higher moments which is not convenient to get bounds on the L^2 norm.

Fortunately one has the trick

$$v\partial_x u - \frac{1}{2}vEu = ve^{-\frac{1}{2}\phi}\partial_x e^{\frac{1}{2}\phi}u$$

where ϕ is the electric potential $\partial_x \phi = -E$. Therefore we pre-multiply the equations by $e^{\frac{1}{2}\phi}$, that is we define

$$w(t, x, v) = e^{\frac{1}{2}\phi(t, x)}u(t, x, v) \text{ and } F(t, x) = e^{\frac{1}{2}\phi(t, x)}E(t, x).$$

These quantities satisfy

$$(5.3) \quad \begin{cases} \partial_t w + v\partial_x w & + vM^{\frac{1}{2}}F & - E\partial_v w & = \frac{1}{2}\partial_t \phi w, & t > 0, & (x, v) \in I \times \mathbb{R}, \\ \partial_t F & - \int wvM^{\frac{1}{2}}dv & & = \frac{1}{2}\partial_t \phi F, & t > 0, & x \in I. \end{cases}$$

The quadratic norm is

$$\mathcal{E}(t) = \int_I \int_{\mathbb{R}} w^2 dv dx + \int_I F^2 dx.$$

PROPOSITION 23. *Solutions of (5.3) are endowed with the energy identity*

$$(5.4) \quad \frac{d}{dt} \mathcal{E}(t) = \int \partial_t \phi \int w^2 dv dx + \int \partial_t \phi F^2 dx$$

where ϕ is bounded by

$$(5.5) \quad \|\partial_t \phi(t)\|_{L^\infty(I)} + \|\partial_x \phi(t)\|_{L^\infty(I)} \leq C \min \left(1, \mathcal{E}(t)^{\frac{1}{2}} \right),$$

and C is a universal constant.

Proof. The proof of (5.4) is immediate. It remains to prove the inequality.

• The electric field E is bounded uniformly in L^2 for the non linear Vlasov Poisson (1.2) using (1.4-1.5). A classical consequence is the following [14]: first one has $\int v f dv \in L^{\frac{3}{2}}(I)$ uniform in time using the energy identity and the L^∞ bound on f . Indeed

$$\left| \int_{\mathbb{R}} v f dv \right| \leq R \left| \int_{|v| < R} f dv \right| + \frac{1}{R} \left| \int_{|v| > R} v^2 f dv \right| \leq 2R^2 \|f_0\|_{L^\infty} + \frac{1}{R} \left| \int_{\mathbb{R}} v^2 f dv \right|.$$

The optimal choice $R = \|f_0\|_{L^\infty}^{-\frac{3}{2}} \left| \int_{\mathbb{R}} v^2 f dv \right|^{\frac{1}{3}}$ yields

$$\left| \int_{\mathbb{R}} v f dv \right| \leq 3 \|f_0\|_{L^\infty}^{\frac{1}{3}} \left| \int_{\mathbb{R}} v^2 f dv \right|^{\frac{2}{3}}$$

which gives after integration in space $\|\int_{\mathbb{R}} v f dv\|_{L^{\frac{3}{2}}(I)} \leq 3\|f_0\|_{L^\infty}^{\frac{1}{3}} (\int_I \int_{\mathbb{R}} v^2 f f x f v)^{\frac{2}{3}}$. Since the L^∞ norm and the total energy are bounded (1.4-1.4), it yields the uniform estimate

$$\left\| \int_{\mathbb{R}} v f(t) dv \right\|_{L^{\frac{3}{2}}(I)} \leq C, \quad \forall t > 0.$$

Therefore $\partial_x(\partial_t \phi)(t) \in L^{\frac{3}{2}}(I)$ uniformly in time. By integration in space one obtains $\partial_t \phi(t) \in L^\infty(I)$ uniform in time. This is the same for $\partial_x \phi = E$, which yields that $\partial_x \phi(t) \in L^\infty(I)$. By integration on the periodic domain one gets that $\phi(t)$, $\partial_t \phi(t)$ and $\partial_x \phi(t)$ bounded in $L^\infty(I)$ uniformly in time. It yields

$$\|\partial_t \phi(t)\|_{L^\infty(I)} + \|\partial_x \phi(t)\|_{L^\infty(I)} \leq C$$

which is one part of the inequality (5.5).

- The remaining part is obtained as follows. The Gauss equation writes

$$\partial_x E = - \int_{\mathbb{R}} u M^{\frac{1}{2}} dv.$$

Since $\left| \int_{\mathbb{R}} u M^{\frac{1}{2}} dv \right| \leq C \left(\int_{\mathbb{R}} u^2 \right)^{\frac{1}{2}}$ for a universal constant, one gets that

$$\|\partial_x E\|_{L^\infty(I)} \leq C \|u(t)\|_{L^2(I \times \mathbb{R})}.$$

The electric potential being bounded in L^∞ : $\|u(t)\|_{L^2(I \times \mathbb{R})} \leq C \|w(t)\|_{L^2(I \times \mathbb{R})}$ It shows that $\partial_x \phi(t) = E(t)$ is indeed bounded in $L^\infty(I)$ by $C\mathcal{E}(t)^{\frac{1}{2}}$. Starting from

$$\partial_x \partial_t \phi = \partial_t E = \int_{\mathbb{R}} uv M^{\frac{1}{2}} dv$$

one gets the same result for the time derivative $\|\partial_x \phi(t)\|_{L^\infty(I)}$ which is also bounded by $C\mathcal{E}(t)^{\frac{1}{2}}$. The proof is ended. \square

We now detail an easy consequence of this estimate.

PROPOSITION 24. *Consider an initial data (u_0, E_0) (resp. (w_0, F_0)) which is small in quadratic norm. Then the solution of (5.2) (resp. (5.3) remains small in quadratic norm for a long time.*

*More precisely one has $\mathcal{E}(t) \leq 2\mathcal{E}(0)$ for all $t \leq T_{**}$ where*

$$T_{**} = O\left(\mathcal{E}(0)^{-\frac{1}{2}}\right).$$

REMARK 25. *This estimate gives some complementary informations about the stability of BGK waves discussed in [12]. Since our quadratic functional setting is slightly different from the one used in this reference, it questions the existence of BGK waves in our setting, together with the possibility of using the integro-differential L to get understanding of non linear Landau damping [13]. This is left for further research.*

Proof. An immediate bound for $\alpha(t) = \mathcal{E}(t)^{\frac{1}{2}}$ is $\frac{d}{dt}\alpha(t) \leq C \min(\alpha(t), \alpha^2(t))$, which turns into

$$\alpha(t) \leq \alpha(0) + C \int_0^t \min(\alpha(s), \alpha^2(s)) ds.$$

The discussion is as follows.

- Assume $\alpha(0) \geq 1$ that is $\min(\alpha(0), \alpha^2(0)) = \alpha(0)$. Then $\alpha(t) \leq e^{Ct}\alpha(0)$ using the Gronwall lemma.
- Assume on the contrary that $\alpha(0) < 1$. Then one has $\alpha(t) \leq g(t)$ for $g(t) = \alpha(0) + C \int_0^t \alpha^2(s)ds$ for t under some threshold value T^* . Since $g'(t) = C\alpha^2(t) \leq Cg^2(t)$ one gets that $g(t) \leq \frac{\alpha(0)}{1-C\alpha(0)t}$. So if $t \leq T_* = \frac{1}{C\alpha(0)}$ the inequality yields

$$\alpha(t) \leq \frac{\alpha(0)}{1 - C\alpha(0)t}.$$

- The general solution is obtained by choosing the worst case. For initial data smaller than 1 which is the interesting situation, a second threshold time

$$T_{**} = \frac{1}{\alpha(0)} - 1.$$

One gets that if $t \leq T_{**}$ then $\alpha(t) \leq \frac{\alpha(0)}{1-C\alpha(0)t}$.
The proof is ended by noticing that for small initial data

$$T_{**} \approx \frac{1}{\alpha(0)} = \mathcal{E}(0)^{-\frac{1}{2}}.$$

□

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